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Theoretical Basics of Rebalancing Alpha in General Portfolio Rebalancing Strategies

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Abstract

This study aims to establish a theoretical framework for "rebalancing Alpha," a metric designed to systematically evaluate the performance differential between rebalanced and non-rebalanced portfolios. The proposed theory is applicable to both pure long and long-short portfolios. By synthesizing existing definitions and providing supplementary proofs, this research consolidates the concept of rebalancing Alpha and offers a systematic performance assessment. The paper further develops a mathematical model for calculating rebalancing Alpha in general portfolio contexts. This metric serves as a key indicator for assessing rebalancing strategies; a higher probability of non-negative rebalancing Alpha indicates a greater likelihood that the rebalancing strategy will generate premium returns. Estimation of approximate values suggests that rebalancing Alpha is correlated with investment cycles. Consequently, rebalancing Alpha provides an explanation for the superior long-term performance of rebalanced portfolios over their non-rebalanced counterparts, thereby offering rigorous theoretical support for portfolio rebalancing theory.

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Keywords

Rebalanced portfolios, long-short portfolios, pure long portfolios, rebalancing Alpha, rebalancing strategies, risk management, investment cycle.

Introduction

In long-term investments, investors usually divide the longer investment period into many small investment cycles, and dynamically adjust the weight of each asset in each small investment cycle to reduce risk because of the weight drift. In addition to risk management, investors also expect to achieve excess returns, namely rebalancing Alpha, through portfolio rebalancing [Kiskiras, Nardon, 2013; Qian, 2018]. For example, the 2008 financial crisis caused stocks to significantly underperform bonds. However, stocks rebounded strongly in 2009, and the portfolio that was rebalanced at the end of 2008 would have achieved better returns. This shows that it is possible to obtain such excess returns, but it also confirms that the existence of rebalancing Alpha is probably very short.

On the question of whether portfolio rebalancing can generate rebalancing Alpha, many scholars have conducted research on different asset types. For example, Qian defines rebalancing alpha as the difference between the returns of a fixed-weight portfolio and its corresponding buy-and-hold portfolio with the same initial weights [Qian, 2018]. In fact, because it is difficult to accurately estimate the return difference between a rebalanced portfolio and a non-rebalanced portfolio, many scholars mistakenly attribute excess returns to the rebalancing strategy [Arnott, Li, Linnainmaa, 2024; Cuthbertson et al., 2015]. Some scholars believe that portfolio growth for both rebalanced and non-rebalanced portfolios is entirely explained by portfolio volatility and that there is no additional contribution to expected growth from rebalancing [Cuthbertson et al., 2015]. However, a mathematical theory can be used to approximate the value of rebalancing Alpha, and to determine the likelihood of its positive or negative value to compare the returns of rebalanced and non-rebalanced portfolios. Therefore, this paper analyzes pure long portfolios and long-short portfolios and provides a mathematical theoretical basis and approximate value for their rebalancing Alpha.

The performance of rebalancing Alpha

Given a fixed-weight portfolio, compare it to a corresponding buy-and-hold strategy. Based on the volatility effect and the return effect, the rebalancing Alpha can be defined as the volatility effect minus the return effect [Qian, 2014; Qian, 2018]. In a pure long portfolio, the volatility effect is non-negative. The return effect is also non-negative and the return effect is 0 only when the geometric mean returns on all assets are equal. At this time, the geometric mean return on the fixed-weight portfolio is greater than the geometric mean return on the buy-and-hold portfolio, and we will get a positive rebalancing Alpha. When the volatility effect is 0, we will get a non-positive rebalancing Alpha, and the rebalancing Alpha is equal to 0 if and only if the return efficiency is 0. The rebalancing Alpha is not always non-negative.

In a long-short portfolio, the imprecise definition of the signs of the volatility effect and the return efficiency will lead to the failure of the rebalancing strategy. For example, in a portfolio of multiple assets, only one asset has a positive weight, and the weights of the other assets are negative. Suppose that we have N assets, the weight of asset 1 is w_1, and the weight of other assets are $w_2, \cdots, w_N$, where $w_1 > 0$ and $w_i < 0$ $(i = 2, \cdots, N)$.

The weighted average of geometric mean returns is $\bar{g} = \sum_{i=1}^{N} w_i g_i$, where g_i $(i = 1, 2, \cdots, N)$ is the geometric mean return on the asset i. Construct a following sum within M investment cycles and exploring Jensen's inequality [Chandler, 1987; Rudin, 1987].

$$(1+\bar{g})^M-\sum_{i=2}^{N} w_i(1+g_i)^M = w_1\left[\frac{1}{w_1}(1+\bar{g})^M+\sum_{i=2}^{N}\frac{-w_i}{w_1}(1+g_i)^M\right]$$

$$\geq w_1\left[1+\frac{1}{w_1}\bar{g}+\sum_{i=2}^{N}\frac{-w_i}{w_1}g_i\right]^M$$

$$\geq w_1(1+g_1)^M$$

Hence, we finally get

$$(1+\bar{g})^M \geq \sum_{i=1}^{N} w_i(1+g_i)^M = (1+g_{BH})^M$$

g_{BH} is the geometric mean return on the buy-and-hold portfolio. This means the geometric mean return on the buy-and-hold portfolio is less than or equal to the weighted average of geometric mean returns, which results in a non-positive return effect. Because return effect equals to the geometric mean on the buy-and-hold portfolio minus the weighted average of geometric mean returns. Suppose that the return rate of the fixed-weight portfolio is defined as g_{FW},

$$(1+g_{FW})^M = \prod_{j=1}^{M}\left(1+\sum_{i=1}^{N} w_i r_{ij}\right)$$

$$= \prod_{j=1}^{M}[\sum_{i=1}^{N} w_i(1+r_{ij})]$$

Hence, we get

$$1+g_{FX} = \sqrt[M]{\prod_{j=1}^{M}[w_1(1+r_{1j})-\sum_{i=2}^{N}(-w_i)(1+r_{ij})]}$$

Where r_{ij} is the return on asset i in the j-th cycle. By Cauchy–Schwarz inequality, we have the following inequality [Choi, 2016; Puntanen, Styan, Isotalo, 2011].

$$\sqrt[M]{\prod_{j=1}^{M}[w_1(1+r_{1j})-\sum_{i=2}^{N}(-w_i)(1+r_{ij})]}+\sqrt[M]{\prod_{j=1}^{M}\left[\sum_{i=2}^{N}(-w_i)(1+r_{ij})\right]}$$

$$\leq \sqrt[M]{\prod_{j=1}^{M} w_1(1+r_{1j})}$$

The above inequality holds if and only if the portfolio always has the capability to repay. Hence,

$$1 + g_{FX} - \sum_{i=2}^{N} w_i(1 + g_i) \leq w_1(1 + g_1)$$

Then we get

$$g_{FX} \leq \sum_{i=1}^{N} w_i g_i = \bar{g}$$

This means the return on the fixed-weight portfolio is less than or equal to the weighted average of geometric mean returns, which results in a non-positive volatility effect.

It has been proved above that using the Jensen's inequality and Cauchy–Schwarz inequality, both the volatility effect and the return effect are non-positive. However, to know whether the rebalancing Alpha is positive or negative, it is necessary to know the specific values of the volatility effect and the return effect. Some rules can be concluded.

Table 1-Comparison between pure long portfolios and long-short portfolios

Parameters \ Portfolios		Pure long portfolios	Long-short portfolios
Volatility effect		≥ 0	≤ 0
Return effect		≥ 0	≤ 0
Rebalancing Alpha	High Volatility effect, low return effect	> 0	
	Low volatility effect, high return effect	< 0	
	Equal volatility effect and return effect	$= 0$	

To further explore the performance of rebalancing Alpha, we need to approximate the value of rebalancing Alpha. The geometric mean return can be approximated by $g = \mu - \frac{\sigma^2}{2}$, where μ is the arithmetic mean return. The geometric mean return on the fixed-weight portfolio is

$$g_{FX} = \bar{g} + \frac{1}{2}\left(\sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_{FW}^2\right)$$

The volatility effect thus is

$$e_v = \frac{1}{2}\left(\sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_{FW}^2\right)$$

To calculate the return effect, we consider the general case. In any portfolio, divide assets into two groups: the long assets group: $w_i > 0$ ($i = 1, 2, \cdots, p$) and the short assets group: $w_i < 0$ ($i = p + 1, \cdots, N$), then the weighted average of geometric mean returns on long assets group is

$$\bar{g}_p = \frac{\sum_{i=1}^{p} w_i g_i}{\sum_{i=1}^{p} w_i} = \frac{\bar{g} - \sum_{i=p+1}^{N} w_i g_i}{\sum_{i=1}^{p} w_i}$$

Exploring the Taylor expansion of $(1 + g_i)^M$ and using $(1 + \bar{g})^M$ to approximate $(1 + g_i)^M = (1 + \bar{g})^M + M(g_i - \bar{g})(1 + \bar{g})^{M-1} + \frac{M(M-1)}{2}(g_i - \bar{g})^2(1 + \bar{g})^{M-2} + O(g_i^3)$. Only consider the long assets group, we have the following result.

$$\left(1+g_{BH,p}\right)^M=\frac{1}{\sum_{i=1}^p w_i}\sum_{i=1}^p w_i(1+g_i)^M$$

$$\approx\left(1+\bar{g}_p\right)^M+M\left(1+\bar{g}_p\right)^{M-1}\frac{\sum_{i=1}^p w_i\left(g_i-\bar{g}_p\right)}{\sum_{i=1}^p w_i}+\frac{M(M-1)}{2}\left(1+\bar{g}_p\right)^{M-2}\frac{\sum_{i=1}^p w_i\left(g_i-\bar{g}_p\right)^2}{\sum_{i=1}^p w_i}$$

$$=(1+\bar{g}_p)^M[1+\frac{M(M-1)}{2\left(1+\bar{g}_p\right)^2}\frac{\sum_{i=1}^p w_i\left(g_i-\bar{g}_p\right)^2}{\sum_{i=1}^p w_i}]$$

$$=(1+\bar{g}_p)^M[1+\frac{M(M-1)}{2\left(1+\bar{g}_p\right)^2}Var(g_p)]$$

$$\approx\left(1+\bar{g}_p\right)^M\left[1+\frac{(M-1)}{2\left(1+\bar{g}_p\right)^2}Var\left(g_p\right)\right]^M$$

$$=\left[1+\bar{g}_p+\frac{(M-1)}{2(1+\bar{g}_p)}Var\left(g_p\right)\right]^M$$

Hence, we have

$$g_{BH,p}=\bar{g}_p+\frac{(M-1)}{2(1+\bar{g}_p)}Var\left(g_p\right)$$

Where $g_{BH,p}$ is the geometric mean return on the buy-and-hold portfolio of the long assets group. According to the above derivation, the return effect of the long assets group can be defined as

$$e_{r,p}=g_{BH,p}-\bar{g}_p=\frac{(M-1)}{2(1+\bar{g}_p)}Var\left(g_p\right)$$

$$Var\left(g_p\right)=\frac{\sum_{i=1}^p w_i\left(g_i-\bar{g}_p\right)^2}{\sum_{i=1}^p w_i}$$

Because of $\bar{g}=(\sum_{i=1}^p w_i)\bar{g}_p+\sum_{i=p+1}^N w_i g_i$. Construct the following formula.

$$\frac{1}{\sum_{i=1}^p w_i}(1+\bar{g})^M-\frac{\sum_{i=p+1}^N w_i}{\sum_{i=1}^p w_i}(1+g_i)^M=[1+\bar{g}_p+\frac{(M-1)}{2\left(1+\bar{g}_p\right)}Var(g_n)]^M$$

$$Var(g_n)=\frac{\left(\bar{g}-\bar{g}_p\right)^2}{\sum_{i=1}^p w_i}-\frac{\sum_{i=p+1}^N w_i\left(g_i-\bar{g}_p\right)^2}{\sum_{i=1}^p w_i}$$

$$e_{r,n}=\frac{(M-1)}{2(1+\bar{g}_p)}Var(g_n)$$

Combining the above two formulas, we can get the following result.

$$(1+g_{BH})^M=\sum_{i=1}^N w_i(1+g_i)^M$$

$$= (1+\bar{g})^M + \sum_{i=1}^{p} w_i (1+g_i)^M - (1+\bar{g})^M + \sum_{i=p+1}^{N} w_i\,(1+g_i)^M$$

$$= (1+\bar{g})^M + (\sum_{i=1}^{p} w_i)\left[1+\bar{g}_p + e_{r,p}\right]^M - (\sum_{i=1}^{p} w_i)\left[1+\bar{g}_p + e_{r,n}\right]^M$$

Because the value of both $e_{r,p}$ and $e_{r,n}$ are very small, we can approximate as follows.

$$(1+g_{BH})^M \approx (1+\bar{g})^M + (\sum_{i=1}^{p} w_i) M\left(e_{r,p} - e_{r,n}\right)\left(1+\bar{g}_p\right)^{M-1}$$

$$\approx [(1+\bar{g}) + \frac{\left(1+\bar{g}_p\right)^{M-1}}{(1+\bar{g})^{M-1}} (\sum_{i=1}^{p} w_i)\left(e_{r,p} - e_{r,n}\right)]^M$$

Hence

$$e_r = \frac{\left(1+\bar{g}_p\right)^{M-1}}{(1+\bar{g})^{M-1}} (\sum_{i=1}^{p} w_i)\left(e_{r,p} - e_{r,n}\right)$$

Because

$$Var\left(g_p\right) - Var(g_n) = \frac{\sum_{i=1}^{N} w_i\left(g_i - \bar{g}_p\right)^2}{\sum_{i=1}^{p} w_i} - \frac{\left(\bar{g} - \bar{g}_p\right)^2}{\sum_{i=1}^{p} w_i}$$

$$= \frac{\sum_{i=1}^{N} w_i\,(g_i - \bar{g})^2}{\sum_{i=1}^{p} w_i} = \frac{Var(g)}{\sum_{i=1}^{p} w_i}$$

Then, we have

$$e_r = \frac{(M-1)\left(1+\bar{g}_p\right)^{M-2}}{2(1+\bar{g})^{M-1}} Var(g)$$

The rebalancing Alpha thus is

$$Rebalancing\ Alpha = e_v - e_r$$

$$= \frac{1}{2}\left(\sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_{FW}^2\right) - \frac{(M-1)\left(1+\bar{g}_p\right)^{M-2}}{2(1+\bar{g})^{M-1}} Var(g)$$

This is an approximate result, but it provides theoretical support for some practical problems.

Conclusion

Rebalancing Alpha can be viewed as a metric used to evaluate rebalancing strategies. The greater the probability that rebalancing Alpha is non-negative, the greater the probability that the rebalancing strategy will generate premium returns. Based on our estimated approximations, we are confident that rebalancing Alpha is correlated with the investment horizon. Therefore, rebalancing Alpha explains why rebalanced portfolios often perform differently from non-rebalanced portfolios over the long term.

Even if this performance may not be ideal, it still provides investors with the potential to utilize rebalancing strategies to achieve their desired objectives.

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Теоретические основы ребалансировки альфа в общих стратегиях управления портфелем

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Аннотация

Данное исследование направлено на формирование теоретической основы для «ребалансировки альфа» – метрики, предназначенной для систематической оценки разницы в доходности между ребалансируемыми и неребалансируемыми портфелями. Предлагаемая теория применима как к чисто длинным, так и к длинно-коротким портфелям. Обобщая существующие определения и предоставляя дополнительные доказательства, данное исследование консолидирует концепцию ребалансировки альфа и предлагает системную оценку эффективности. В статье также разработана математическая модель для расчета ребалансировки альфа в общем контексте портфеля. Эта метрика служит ключевым показателем для оценки стратегий ребалансировки; более высокая вероятность неотрицательной ребалансировки альфа указывает на большую вероятность того, что стратегия ребалансировки принесет премиальную доходность. Оценка приблизительных значений предполагает, что ребалансировка альфа коррелирует с инвестиционными циклами. Следовательно, ребалансировка альфа объясняет, почему ребалансируемые портфели в долгосрочной перспективе часто показывают иные результаты по сравнению с их

неребалансируемыми аналогами, тем самым предлагая строгую теоретическую поддержку теории ребалансировки портфеля.

Для цитирования в научных исследованиях

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Ключевые слова

Ребалансируемые портфели, длинно-короткие портфели, чистые длинные портфели, ребалансировка альфа, стратегии ребалансировки, управление рисками, инвестиционный цикл.

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